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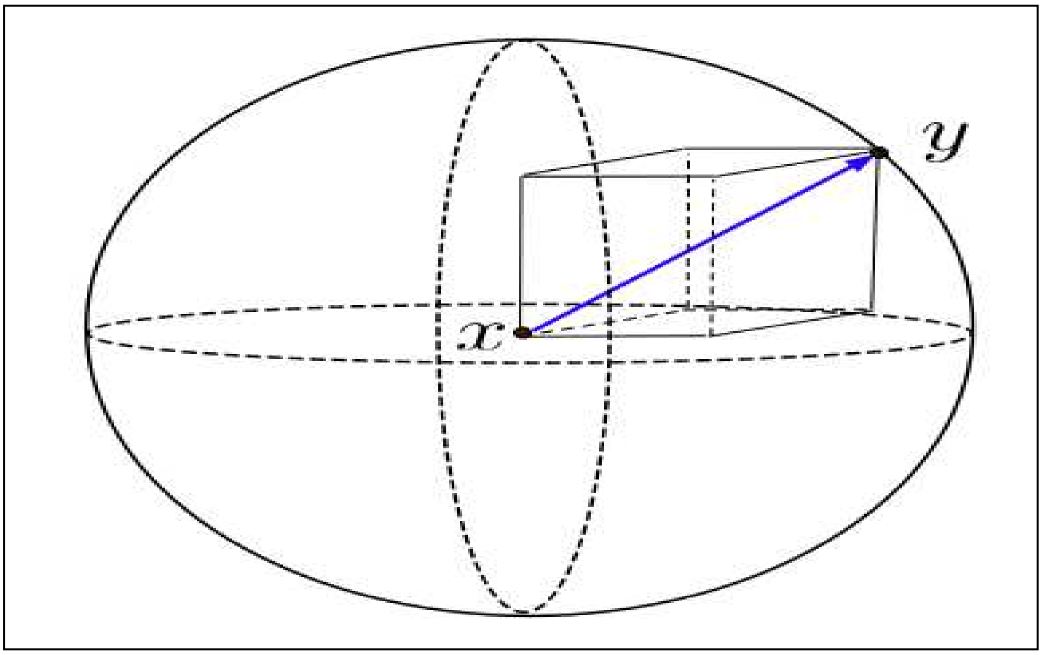
UNIQUE FIXED POINT THEOREMS IN COMPLETE METRIC SPACE



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ABSTRACT

In this paper, some fixed point theorem are proved related to Complete Metric Space, in which the fixed point theorems in A Meir Emmatt [4], Brouwer [5], Caccioppoli [8] and Dolhare U P [1,2,3] as a special cases. By fixed point technique, we find the fixed points which are invariant under the action of functional equation $f(x) = x$ by Generalizing Some Fixed Point Theorems. We prove Unique fixed point theorems for self-maps satisfying some contractive type conditions. In this paper we prove the contraction mapping theorems states that a strict contraction on a complete Metric Space has a unique fixed point . The generalization of contraction mapping in complete metric space which include some fixed point results of Banach , Cric L. B., Kannan ,Sehgal , Edelstein, Das and Gupta , in complete metric space as a special cases

KEY WORDS : Complete Metric Space,Fixed point etc.

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1 INTRODUCTION :

The French Mathematician **H. Poincare** [1854-1912] first recognized the importance of the study of the nonlinear problems. Since then several methods have been developed for proving the fixed point by generalizing the fixed point theorems in Complete Metric Space. The study of fixed points of Selfmappings satisfying contractive conditions which is one of the research activity. Metric fixed point theory used to **Banach** fixed point theorem (1922). Fixed point theorems are applied in various fields of science. The theory started with the generalization of the Banach fixed point theorem using contraction and non expansive mappings **Caccioppoli [8],M.S. Khan [9], S.K. Chatterjea[10], also Rhoades B.E. Cric[13]** worked 0n these mapping.**Dhage, Dolhare U.P**. **and Andrian Petrusel** used non-selfmaps to obtain some fixed point theorems in metric space. **M.S.Khan** proved some interesting results for multivalued continuous mappings. We also generalized the results of **Chtterjee s.**[10] to expansive selfmaps and non-selfmaps.

2. COMPLETE METRIC SPACE :

We need the following definitions for to prove the fixed point theorems.

Definition (2.1):Dolhare U.P.[2] Let (X, d) be a metric space, A sequence $\{x_n\}$ in X is called a Cauchy sequence if for any $\varepsilon > 0$, there is an $n_\varepsilon \in N$ s.t. $d(x_m, x_n) < \varepsilon$ for any $m \geq n_\varepsilon, n \geq n_\varepsilon$

Theorem 2.1 : Any convergent sequence in Metric Space is a Cauchy Sequence.Proof : Consider that $\{x_n\}$ is a sequence which converges to x. Let $\varepsilon > 0$ be given, then there is an $N \in \mathbb{N}$ such that $d(x_n, x) < {}^{\varepsilon}/_{2}$ for all $n \in N$. Let $m, n \in N$ be such that $m \geq N, n \geq N$ then

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x) + d(x_n, x) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Definition 3.2 : Dolhare U.P.[3] Let (X, d) be a metric space. A mapping $f: X \to X$ is called a contraction or contraction mapping if there exists a number $\alpha < 1$ such that

$$d(\, f(x), f(y)\,) \leq \alpha\, d(x, y) \qquad \dots\dots\dots (3.1)$$

For all $x, y \in X$ and α is called the contraction coefficient.

Thus, a contraction maps points closer together in particular, for every $x \in X$, and any $r > 0$ all points y in the ball $B_r(x)$ are map into a ball $B_s(f_x)$ with $s < r$. Some times a map satisfying equation (3.1) with $\alpha = 1$ is also called a **contraction**, and then map satisfying (3.1) with $\alpha < 1$ is called a **strict contraction**. It follows from (3.1) that a contraction mapping is uniformly continuous.

If $f: X \to X$, then a point $x \in X$ such that

$$f(x) = x \qquad \dots\dots\dots (3.2)$$

is called a fixed point of f.

Definition 2.1 :Dolhare U.P.[1] Let X be a set and $f: X \to X$ be a map. A point $x \in X$ is called a fixed point of f if $f(x) = x$.

i.e. If f is defined on the real number by $f(x) = x^2 - 7x + 12$. We know that $x = 3,4$ are roots of the equation. Let us consider $f(x) = x$. Where $f(x) = \dfrac{x^2 + 12}{7}$ then $x = 3$ and $x = 4$ are two fixed points of $f(x)$.

Hence $\{x_n\}$ is a Cauchy sequence.

Definition 2.2 :Dolhare U.P. [3] A Metric Space (X, d) is said to be **complete** if every Cauchy sequence in X converges to a point in X.
Definition 2.3 : Let (X, d) be a Metric Space, A mapping $f: X \to X$ is called a contraction if there exists a number $\alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha\, d(x, y) \qquad \ldots\ldots\ldots 2.1$$

for all $x, y \in X$ and α is a constant.
Definition 2.4 Caccioppoli [8]: A mapping $T: X \to X$ where (X, d) is a Metric Space is said to be a **C-Contraction** if there exists $\alpha \in \left(0, \frac{1}{2}\right)$ such that for all $x, y \in X$, then the following inequality holds

$$d(T_x, T_y) \leq \alpha \left(Cad(x, T_y) + d(y, T_x)\right) \qquad \ldots\ldots\ldots 2.2$$

The concept of C-Contraction was defined by S.K.Chatterjea [8] in 1972 and he proved that if (X, d) is a Complete Metric Space, then every c – contraction on X has a unique fixed point.

Theorem 3.1 : Dolhare U.P.[1] *Let (X, d) be a complete metric space and $T: X \to X$ be satisfying*

$$d(T_x, T_y) \leq \alpha \cdot d(x, y), \quad \forall\, x, y \in X \qquad \ldots\ldots\ldots (3.3)$$

Where $0 \leq \alpha < 1$. Then T has a unique fixed point in X.

Theorem 3.2 : Dolhare U.P.and Bele C.D.[2] *Let f be a self maps of f-orbitally complete D- metric space X satisfying*
$(x, y, z)\rho(C, f_y, f_z) = \lambda\, \rho$ *(where $0 \leq \lambda \leq 1$ then f has a unique fixed point.*
Theorem 3.3 : Dolhare U.P.[1] *:Let (X ,ρ) be a complete metric space and f be a self map on X such that f^2 is continuous if $g : f(x) \to X$ such that g f(x) ⊂ f^2(x) and g (f(x))= f (g (x)) both sides are defined for all x,y ϵ f(x). Then f and g have unique common fixed point.*
In **Khan** (1976), the following fixed point theorem is generalized as follows.
Theorem 2.2 : Khan [9] Let T be a self mapping of a complete metric space (X, d) and satisfying

$$d(T_x, T_y) \leq \alpha \left(d(x, T_y) d(y, T_x)\right)^{1/2} \qquad \ldots\ldots\ldots 2.3$$

for all $x, y \in X$ and $0 \leq \alpha < 1$, then T has a unique fixed point.
Theorem 3.4 : (Sehgal V.M. [12]) Let (X, d) be a complete metric space, and $f: X \to X$ a continuous mapping satisfying the condition ; there exists a $k > 1$ such that for each $x \in X$, there is a positive integer $n(x)$ such that for all $y \in X$

$$d\left(f^{n(x)}(y), f^{n(x)}(x)\right) \leq k\, d(y, x) \qquad \ldots\ldots\ldots (\,)$$

Then f has a unique fixed point u and $f^n(x_0) \to u$ for each $x_0 \in X$.
Definition 2.5: A function $\psi: [0, \infty) \to [0, \infty)$ is called an altering distance if (1) $\psi(0) = 0$ and (2) ψ is continuous function and monotonically non-decreasing.
D.P.Shukla and Ruchira Sing [7] generalized the following theorem.
Theorem 2.3 : Let (X, d) be a complete metric space. Let ψ be an altering function, and Let $f: X \to X$ be a self-mapping which satisfies the following inequality

$$\psi\left(d(f_x, f_y)\right) \leq c\, \psi(d(x, y)) \qquad \ldots\ldots\ldots 2.4$$

for all $x, y \in X$ and $0 < c < 1$, then f has a unique fixed point.

Theorem 1 : [Jaggi D. S.and Jaggi] [11] : Let f be a continuous (t) = 0 if and only if t = 0.self-map defined on a complete metric space (X , d). Further, Let f satisfy the following condition.

$$d(f(x), f(y)) \leq \frac{\propto d(x,f(x))\, d(y,f(y))}{d(x,y)} + \beta\, d(x,y)$$

for all $x, y \in X$, $x \neq y$ and for some $\propto, \beta \in [0,1)$ with $\propto + \beta < 1$, then f has a unique fixed point in X.
D.S. Jaggi generalized theorem 1 for some integer m as follows.
Theorem 2 : [Jaggi D. S.] [11] : Let f be a self-map define on a complete metric space (X , d) such that for some positive integer m, f satisfy the condition

$$d(f^m(x), f^m(y)) \leq \frac{\propto d\left(x,f^m(x)\right).\, d(y,f^m(y))}{d(x,y)} + \beta d(x,y)$$ for all $x, y \in X$, $x \neq y$ and for some $\propto, \beta \in [0,1)$ with

$\propto + \beta < 1$. if f^m is continuous then f has a unique fixed point.
Jungck proved the following theorem for f–contractive point- to- point mapping for fixed point .
Theorem 3 : [Jungck] [13] : Let X be a complete metric space. Let f and g be commuting continuous self-maps on X such that $g(x) \subset f(x)$ further, let there exist a constant $\propto \in (0,1)$ such that for every x , y in X

$$d(g_x, g_y) \leq \propto d(f_x, f_y)$$

then f and g have a unique common fixed point.

Theorem 4 : [Lj. B. Ciric] [14] : Let X be a complete metric space. Let f be a self-map on X such that for some constant $\propto \in (0,1)$ and for every x, y in X.

$$d(f_x, f_y) \leq \propto \max\{d(x,y),\ d(x,f_x),\ d(y, f_y),\ d(x, f_y),\ d(y,f_x)\}$$

Then f possesses a unique fixed point.
In 1980 the Jaggi and Dass [11] generalized some fixed point theorems with the mapping satisfying

$$d(f_x, f_y) \leq \alpha\, d(x,y) + \beta \frac{d(x,f_x)d(y,f_y)}{d(x,y) + d(x,f_y) + d(y,f_x)} \qquad \ldots\ldots\ldots 2.5$$

for all $x, y \in X$ and for $\alpha + \beta < 1$

MAIN RESULTS

We generalized the inequality (2.5) by using (2.1),(2.2) with the help of (2.4) as follows
Theorem : Let (X, d) be a Complete Metric space and Let f be a mapping from X into itself. Suppose that f satisfies the following condition

$$d(f_x, f_y) \leq \left(\frac{d(x,f_y) + d(y,f_x)}{d(x,f_x) + d(y,f_y) + 1}\right) d(x,y) \qquad \ldots\ldots\ldots 2.6$$

for all $x, y \in X$, then f has at least one fixed point x.
Proof : Consider the sequence $\{x_n\}$ such that $x_{n+1} = fx_n$ and let $x_0 \in X$ be arbitrary, then

$$d(x_{n+1}, x_n) = d(fx_n, fx_{n-1})$$

$$\leq \left(\frac{d(x_n,x_n) + d(x_{n-1},x_{n+1})}{d(x_n,x_{n+1}) + d(x_{n-1},x_n) + 1}\right) d(x_n, x_{n-1})$$

$$\leq \left(\frac{d(x_{n-1},x_{n+1})}{d(x_n,x_{n+1}) + d(x_{n-1},x_n) + 1}\right) d(x_n, x_{n-1})$$

$$\leq \left(\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{d(x_n,x_{n+1})+d(x_{n-1},x_n)+1}\right)d(x_n,x_{n-1}) \qquad \ldots\ldots 2.7$$

Consider, $\alpha_n = \dfrac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{d(x_n,x_{n+1})+d(x_{n-1},x_n)+1}$ 2.8

By using (2.7) and (2.8), we have

$$\begin{aligned} d(x_{n+1},x_n) &\leq \alpha_n d(x_n,x_{n-1}) \\ &\leq (\alpha_n\alpha_{n-1})d(x_{n-1},x_{n-2}) \\ &\leq (\alpha_n\alpha_{n-1}\alpha_{n-2})d(x_{n-2},x_{n-3}) \\ &\vdots \\ &\leq (\alpha_n\alpha_{n-1}\ldots\alpha_1)d(x_1,x_0) \end{aligned}$$

Suppose that, $a_p = (\alpha_p,\alpha_{p-1},\ldots,\alpha_1)$

Since, $\lim\limits_{p\to\infty}\dfrac{a_{p+1}}{a_p}=0 \quad i.e. \quad \sum\limits_{p=1}^{\infty} a_p < \infty$

This shows that,

$$\sum_{p=n}^{m-1}(\alpha_p\,\alpha_{p-1}\,\ldots\,\alpha_1)\to 0$$

As $m,n\to\infty$, this shows that $\{x_n\}$ is a Cauchy sequence and which converges to x where $x\in X$. This shows that f has a unique fixed point x. In this paper we are also generalized new rational expression in Complete Metric Space and find out fixed point by generalizing fixed point theorems.

Theorem 3.11 : *Let (X,ρ) be a complete metric space suppose f is contraction mapping and λ is a constant for f and λ^n is the constant for f^n. Then f^n is also a contraction and f^n has a fixed point.*

Proof : Firstly we will show that the theorem is true for $n=2$. Since f is a contraction consider $\lambda<1$, then

$$d(f(x),f(y))\leq\lambda\, d(x,y)$$

We can apply f to $f(x)$ and $f(y)$ such that

$$d(f^2(x),f^2(y))\leq\lambda\, d(f(x),f(y))$$

Since $d(f(x),f(y))\leq\lambda\, d(x,y)$

$$d(f^2(x),f^2(y))\leq\lambda\, d(f(x),f(y))\leq\lambda^2 d(x,y)$$

Thus, $$d(f^2(x),f^2(y))\leq\lambda^2 d(x,y)$$

Since $\lambda<1,\lambda^2<1$ then f^n is a contraction,

Since f^n is a contraction then it is true that f^{n+1} is also contraction

$$d\big(f^{n+1}(x), f^{n+1}(y)\big) \le \lambda^{n+1}\, d\big(f(x), f(y)\big) \le \lambda^{n+1} d(x,y)$$

Thus, $d\big(f^{n+1}(x), f^{n+1}(y)\big) \le \lambda^{n+1} d(x,y)$

Thus by induction the theorem is true for all n. If $f(x) = x$, then

$$f^2(x) = f\big(f(x)\big) = f(x) = x$$

Then $f^n(x) = x$

Hence by induction f^n is contraction and f^n has the unique fixed point.

CONCLUSION : In the present paper we used contraction mapping.By using contraction Mapping we can find out unique fixed points of self maps in complete Metric Space.

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